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# Nonlinear Consequences of Energetic Particle Instabilities

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# Outline

- Examples of Experimental Data
- Near-threshold Technique and Bump-on-tail Model
- Nonlinear Bifurcations and Phase Space Structures
- Generalization to Energetic Particles in Tokamaks
- Multiple Modes and Global Transport
- Concluding Remarks



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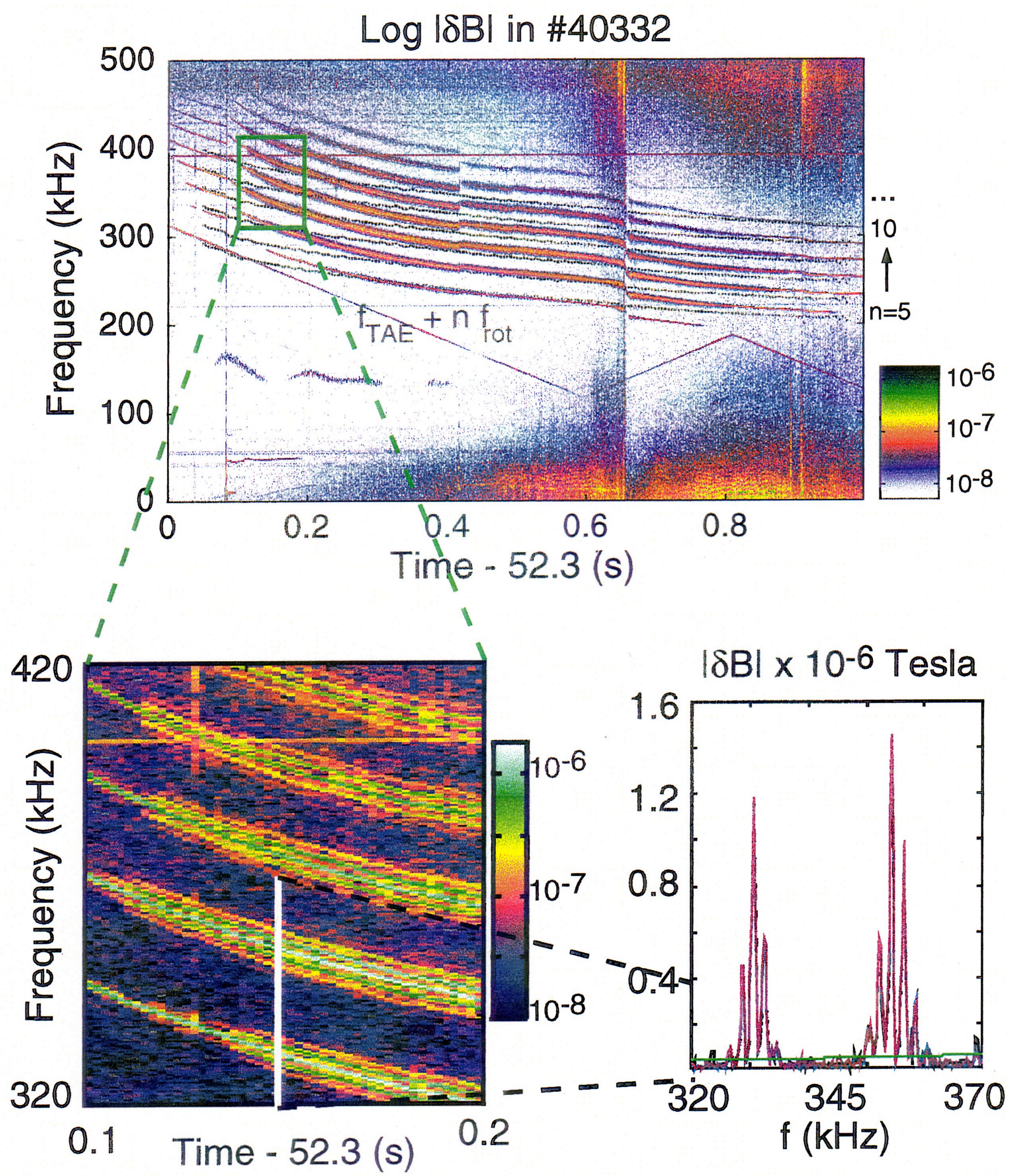
# NONLINEARITY IN ACTION

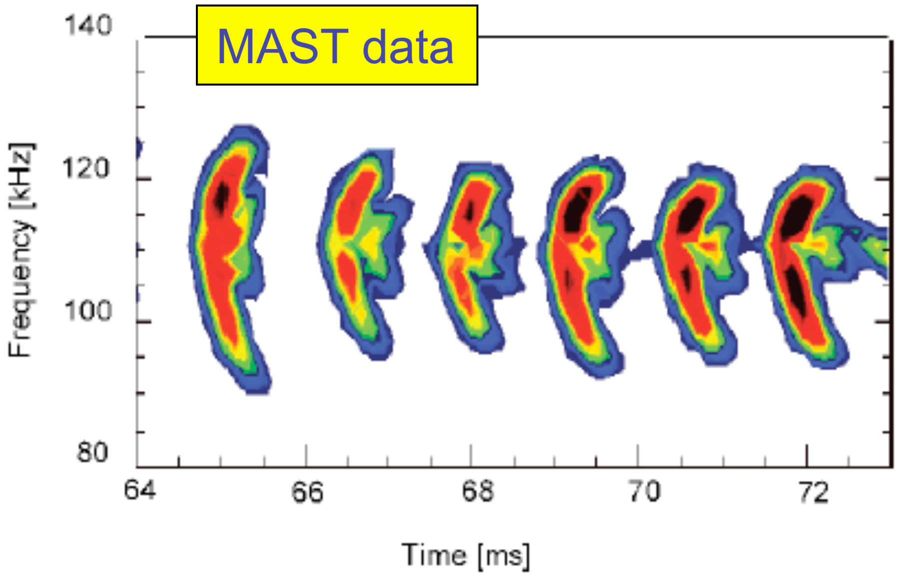
# Nonlinear Splitting of Alfvén Eigenmodes in JET

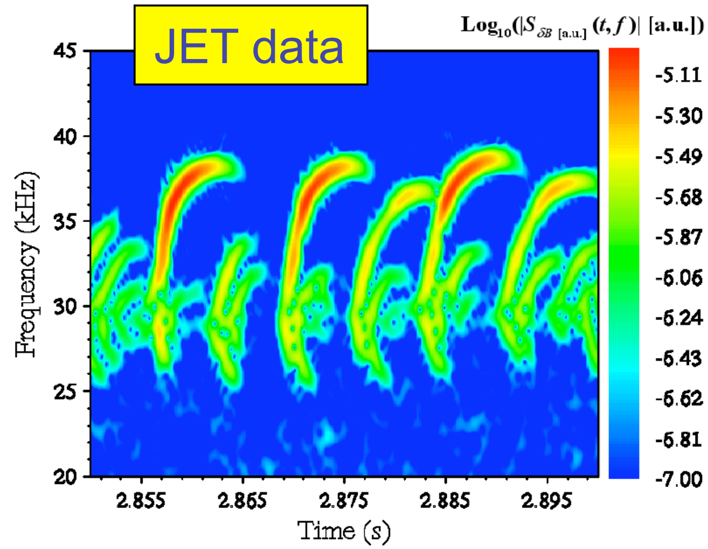


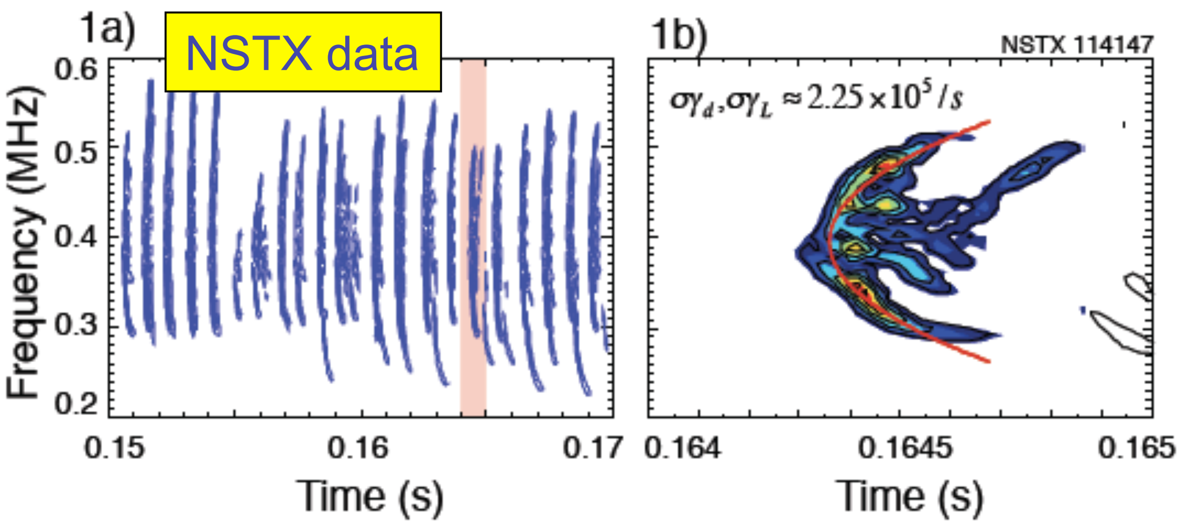

*A. Fasoli, et al., PRL **81**, 5564 (1998)*

*R. F. Heeter, et al., PRL **85**, 3177 (2000)*

# Rapid Frequency Sweeping Events




The ms timescale of these events is much shorter than the energy confinement time in the plasma



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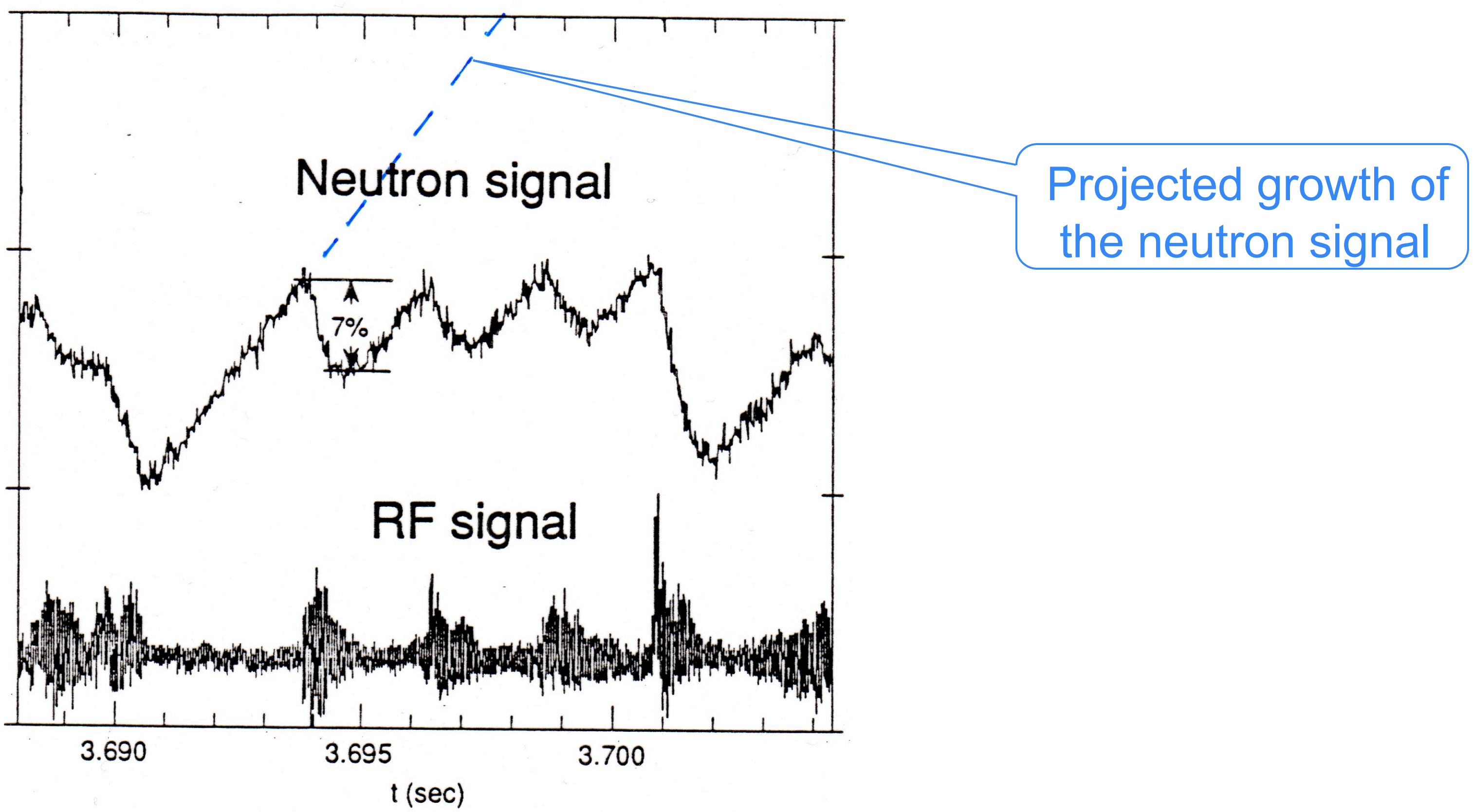
# Alfvén Wave Instability and Particle Loss in TFTR

Saturation of the neutron signal reflects anomalous losses of the injected beams. The losses result from Alfvénic activity.




*K. L. Wong, et al., PRL **66**, 1874 (1991)*



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# THEORY AT THE THRESHOLD

# Near-threshold Nonlinear Regimes

## Why study nonlinear response near the threshold?

– Typically, macroscopic plasma parameters evolve slowly compared to the instability growth time scale

– Near-threshold simulations are intrinsically challenging for most codes

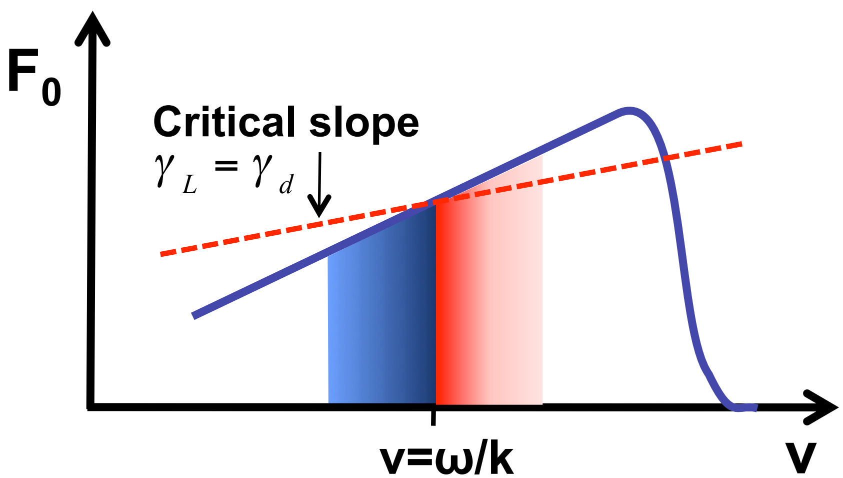
– Identification of the soft and hard nonlinear regimes is crucial to determining whether an unstable system will remain at marginal stability

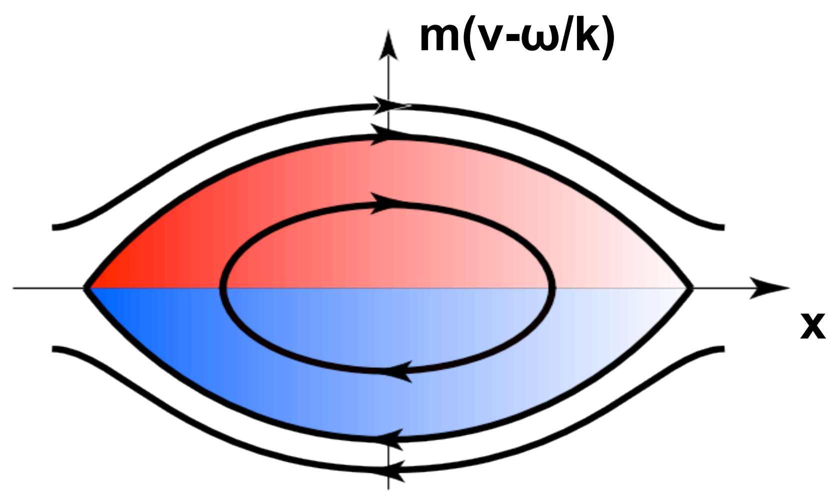
– Long-lived coherent nonlinear structures can emerge

– Multiple modes can keep the system near marginal stability

# Key Ingredients in Theory

- Particle injection and effective collisions, $\nu_{eff}$, create an inverted distribution of energetic particles $F_0(v)$.
- Discrete spectrum of unstable modes.
- Instability drive, $\gamma_L$, due to wave-particle resonance ($\omega$-kv=0).
- Background dissipation rate, $\gamma_d$, determines the critical gradient for the instability.



# The Bump-on-Tail Model Formalism

$$\boxed{\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial \zeta} + \frac{ek}{2m}\left[\hat{E}(t)e^{i\zeta} + \text{c.c.}\right]\frac{\partial F}{\partial u} = \left[\nu^3\frac{\partial^2}{\partial u^2} + \alpha^2\frac{\partial}{\partial u} - \beta\right](F - F_0)}$$

$$\boxed{\frac{\partial \hat{E}}{\partial t} = -4\frac{\omega}{k^2}\pi e\int f_1\,du - \gamma_d\hat{E}}$$

$$u \equiv k\mathrm{v} - \omega$$

$$\zeta \equiv kx - \omega t$$

$$F = F_0 + f_0 + \sum_{n=1}^{\infty}\left[f_n \exp(in\zeta) + c.c.\right]$$

$$E = \frac{1}{2}\left[\hat{E}(t)e^{i\zeta} + \text{c.c.}\right]$$

# Near-threshold Ordering

- The time-scale of interest, $\tau$, is shorter than the trapped particle bounce period:

$$\omega_B \tau \equiv \left(ek\hat{E}/m\right)^{1/2} \tau << 1$$

- Applicability window for near-threshold ordering:

$$\left(\gamma_L - \gamma_d\right)/\gamma_L \leq \left(\omega_B \tau\right)^4 << 1$$

- This ordering can hold indefinitely if the effective collision frequency is greater than the bounce frequency.

- The ensuing ordering for the distribution function:

$$F_0 >> f_1 >> f_0, f_2$$

# Mode Evolution Equation

Near marginal stability the mode amplitude $A$
is governed by the following nonlinear equation:

$$\frac{dA}{d\tau} = A(\tau) - \frac{1}{2}\int_0^{\tau/2} dz\, z^2 A(\tau - z) \int_0^{\tau-2z} dx\, e^{-\hat{\beta}(2z+x) - \hat{\nu}^3 z^2 (2z/3+x) + i\hat{\alpha}^2 z(z+x)} \times A(\tau - z - x) A^*(\tau - 2z - x)$$

Krook factor:

$$\hat{\beta} \equiv \beta / (\gamma_L - \gamma_d)$$

Diffusion factor:

$$\hat{\nu} \equiv \nu / (\gamma_L - \gamma_d)$$

Drag factor:

$$\hat{\alpha} \equiv \alpha / (\gamma_L - \gamma_d)$$

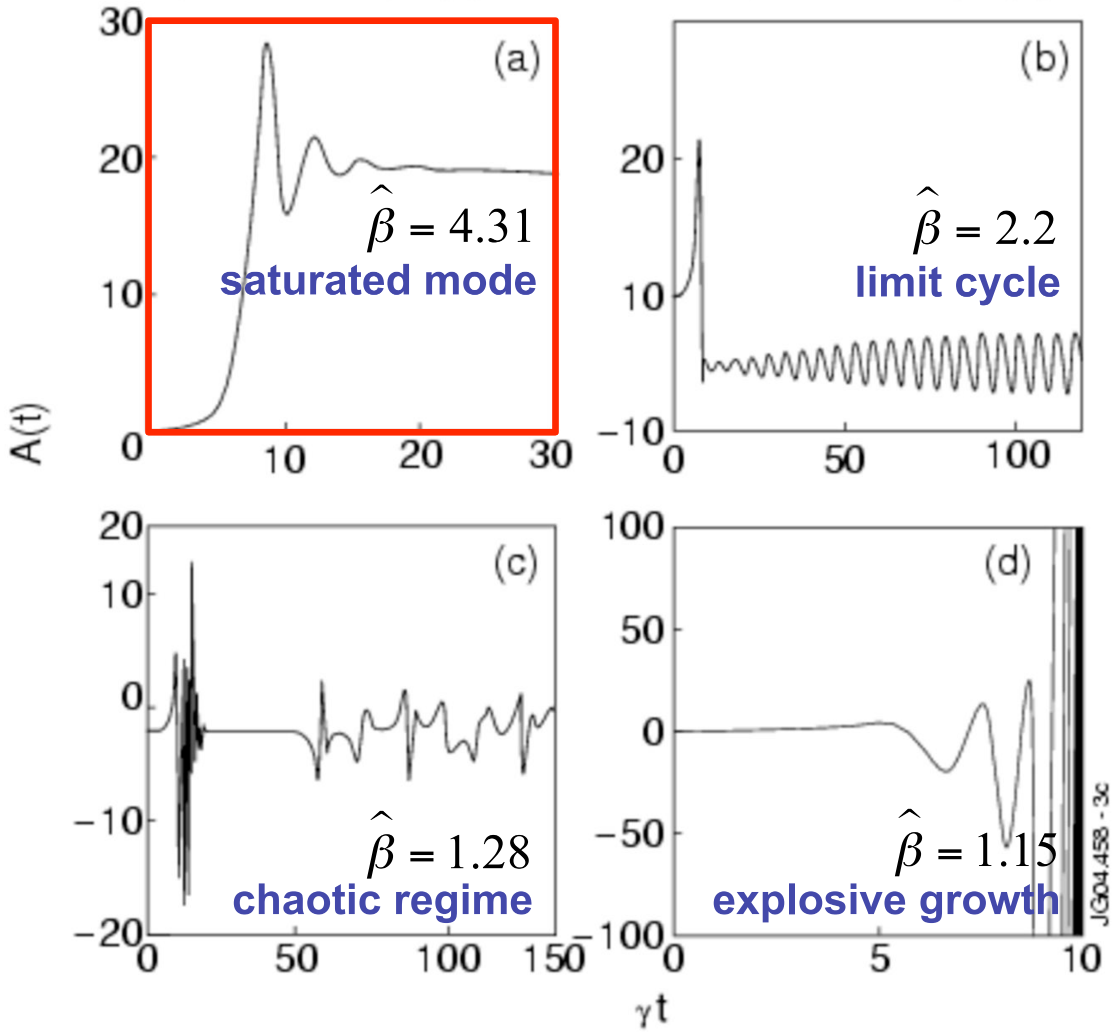
Drag gives oscillatory behaviour, in contrast to the Krook and diffusive cases.


*H.L. Berk et al., PRL **76**, 1256 (1996)*

*B.N. Breizman et al., Phys. Plasmas **4**, 1559 (1997)*

*M.K. Lilley et al., PRL **102**, 195003 (2009)*

# Transition from Steady Nonlinear Saturation to the Explosive Regime (Krook collision case)




**Instability drive increases from (a) to (d)**

$$\hat{\beta} \equiv \frac{\nu_{eff}}{\left(\gamma_L - \gamma_d\right)}$$



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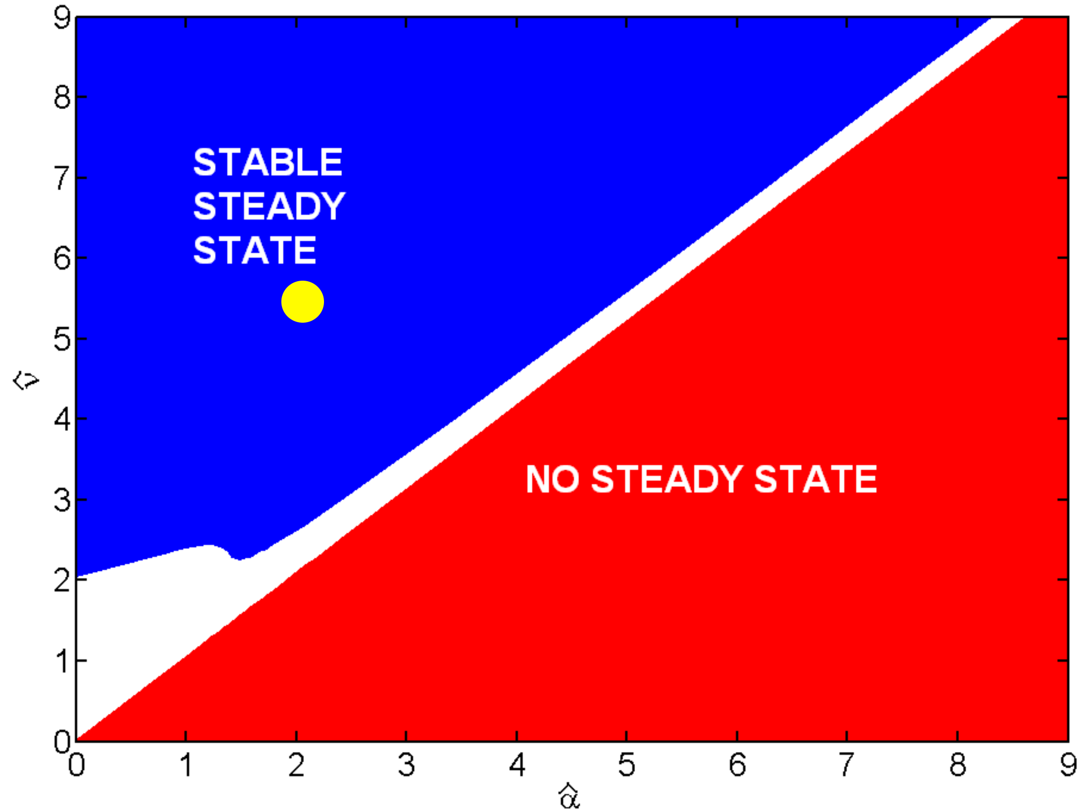
# Mode Saturation Diagram (diffusion+drag)

- For diffusion + weak drag, steady state solution does exist
- For an appreciable amount of drag, this solution becomes unstable (pitchfork splitting )
- The solution is explosive when drag dominates





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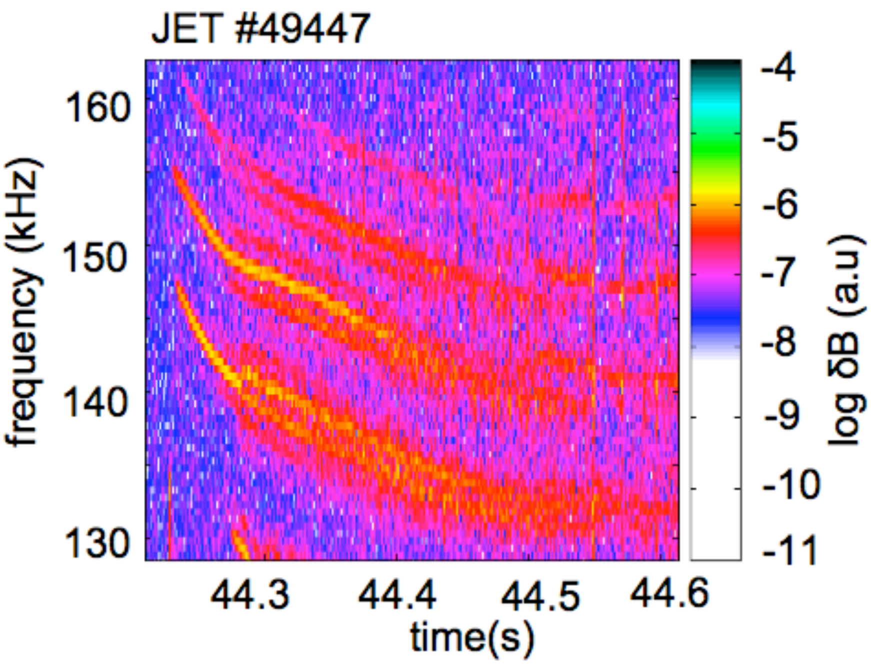
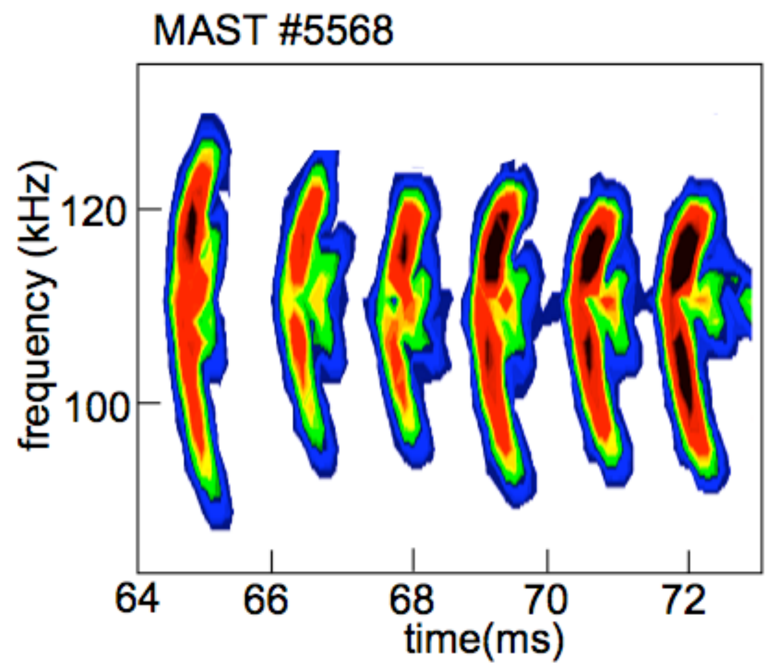
# Fast Particle Driven TAEs

ICRH drive (JET)




*R. F. Heeter, et al., PRL **85**, 3177 (2000)*

NBI drive (MAST)




*R. F. Pinches, et al., PPCF **46**, S47(2004)*



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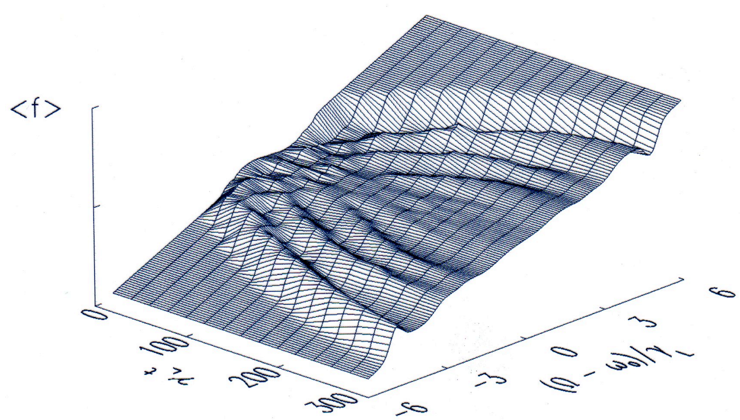
# SPONTANEOUS FREQUENCY SWEEPING

# (phase space holes and clumps)

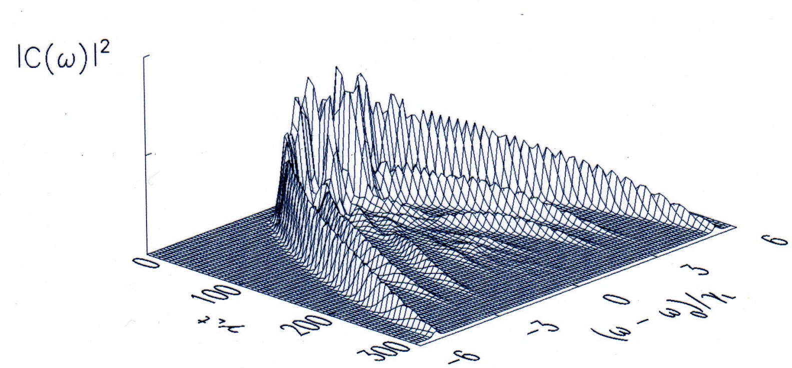
# Spontaneous Chirping of Weakly Unstable Mode

- ❑ Simulation of near-threshold bump-on-tail instability *(N. Petviashvili, 1997)* reveals spontaneous formation of coherent phase space structures (clumps and holes) with time-dependent frequencies.
- ❑ The phase space structures seek lower energy states to compensate energy losses due to background dissipation.
- ❑ Clumps move to lower energies and holes move to higher energy regions.

**Spatially averaged distribution function**

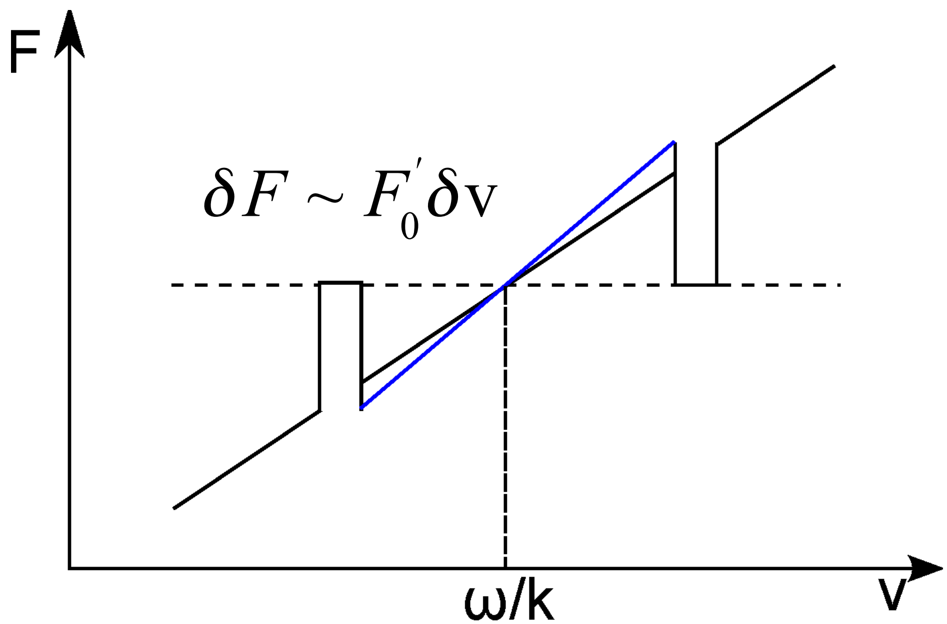



**Mode power spectrum**



# Dynamics of Holes and Clumps at Early Times

- Holes/clumps are the original resonant particles
- They move slowly compared to the bouce period




- The wave amplitude is constant:

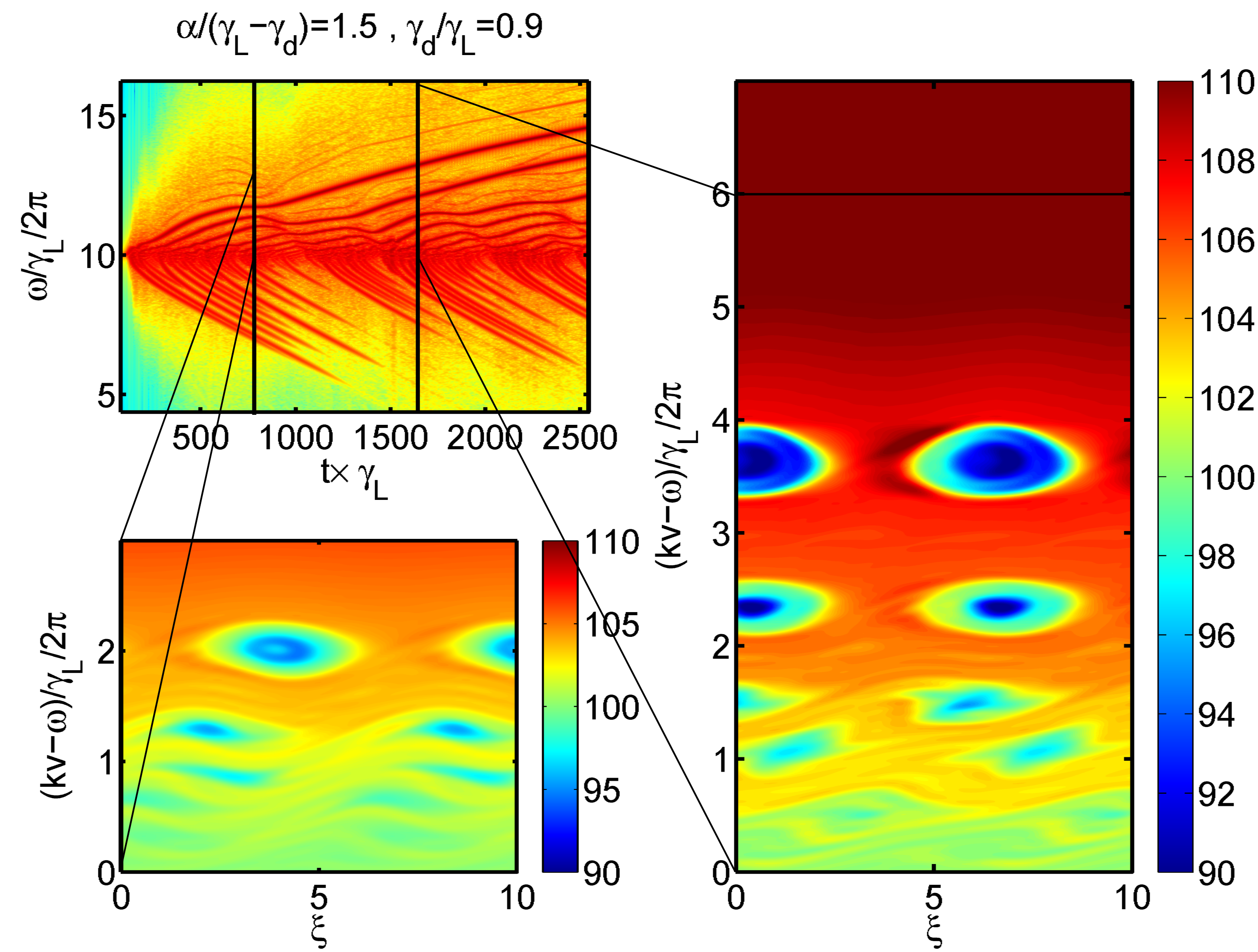
$$\omega_B = \left(16/3\pi^2\right)\gamma_L$$

- Particles cant get inside separatrix.
- Hole/clump gets deeper/ higher as it moves:

$$\delta\omega = \left(16/3\pi^2\right)\gamma_L\sqrt{2\gamma_d t/3}$$

*H.L. Berk, B.N. Breizman, N.V. Petviashvili, Phys. Lett. A* ***234****, 213 (1997)*

# Effect of Drag: Holes Grow Faster, Clumps Decay






*Simulations by M.K. Lilley (2010)*

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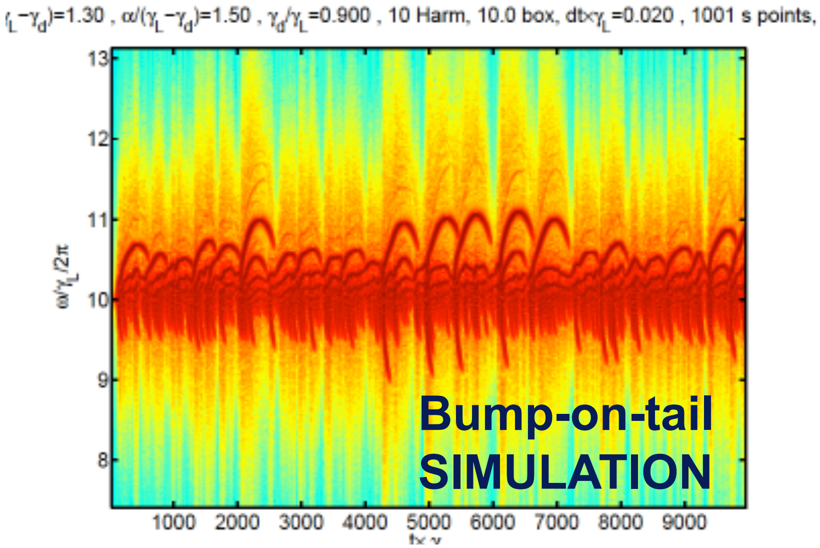
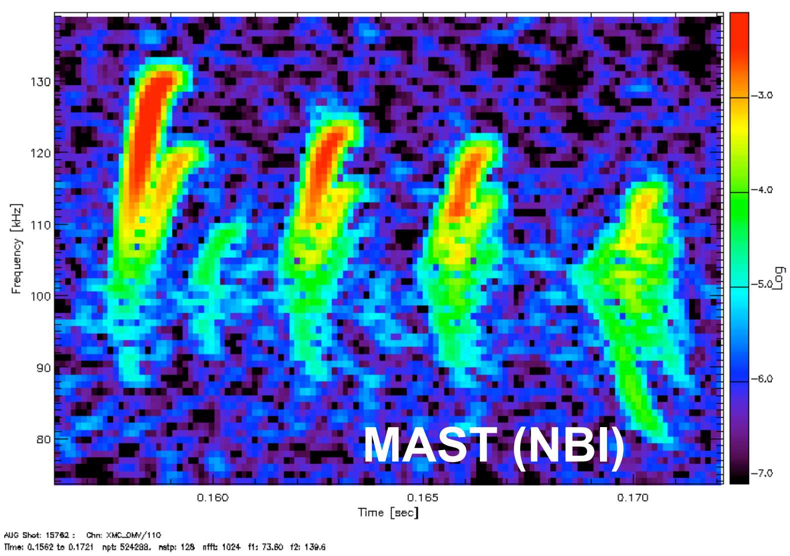
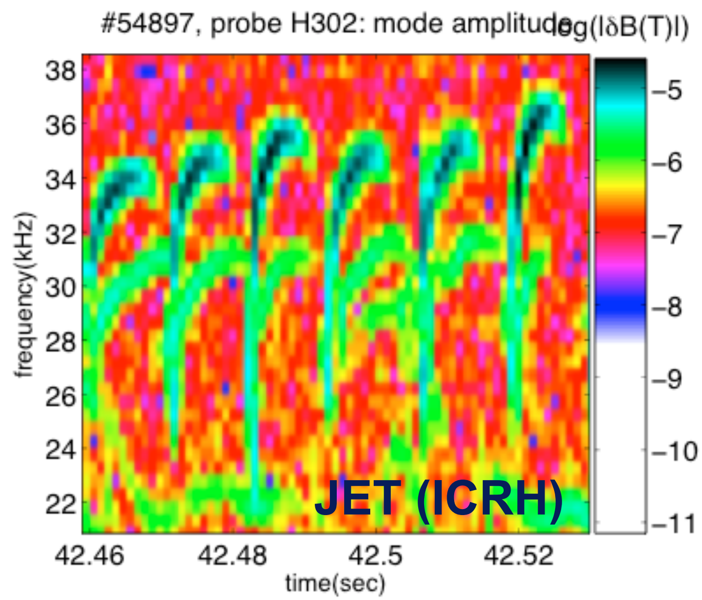
# Drag+Diffusion Give Hooked Frequency Pattern




- Hooked frequency chirp seen in BOT simulations.
- Also seen in MAST (NBI) and JET (ICRH)






*M.K. Lilley , TTF Meeting (2010)*

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# Simple Model for Drag-Diffusion Competition

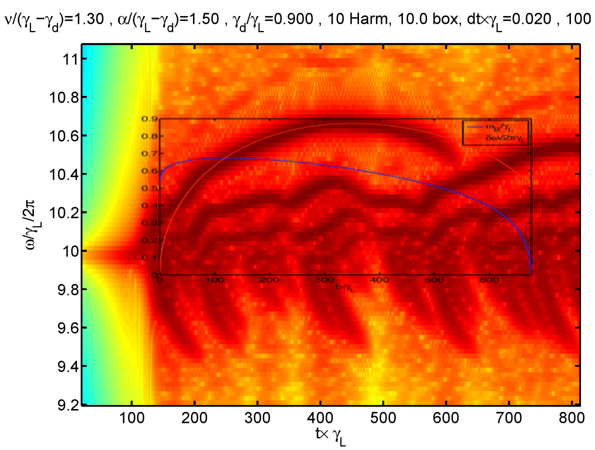
- Poisson equation: $\delta\omega\omega_B = \left(16/3\pi^2\right)g\gamma_L$
- Chirping and drag deepen the hole, diffusion fills it:

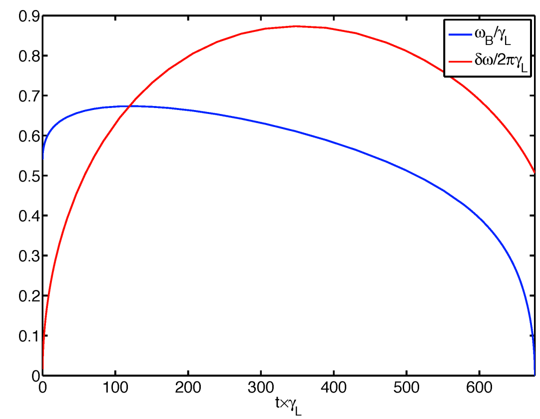
$$dg/dt + g\nu^3/\omega_B^2 = d\delta\omega/dt + \alpha^2$$

- Energy balance equation:

$$\gamma_d\omega_B^3 = 3\left(16/3\pi^2\right)g\gamma_L\left(d\delta\omega/dt + \alpha^2\right)$$

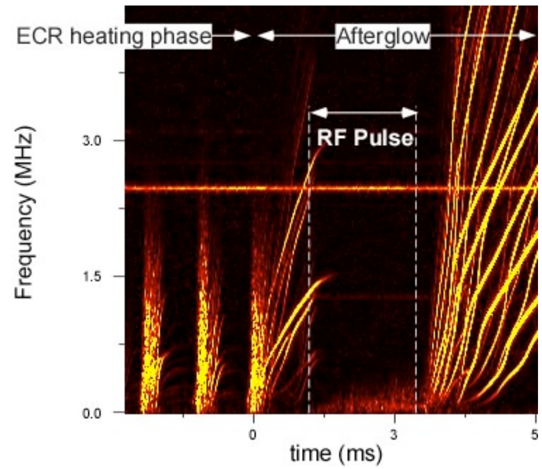
- The model reproduces hooked sweep:



# How to Treat Long-range Frequency Sweeping

***Hot electron interchange modes in Terrella***
***(Courtesy of Michael Mauel, Columbia University)***




## Observation:

Experiments exhibit signals with large (order of unity) frequency sweeping.

## Issue:

How can a small group of particles produce a large change in the mode frequency?

## Proposed physics mechanism:

Initial instability leads to particle trapping and creates a modulated beam of resonant particles (BGK-type structure).

As the beam particles slow down significantly , they produce a signal that deviates considerably from the initial mode frequency.

*B.N. Breizman, Nucl. Fusion, to be published (2010)*



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# Slowly Varying Periodic Electrostatic Wave

- **Wave electrostatic potential (with a spatial period $\lambda$):**

$$\varphi\left[z - s(t);t\right]$$

- **Lab-frame Hamiltonian:**

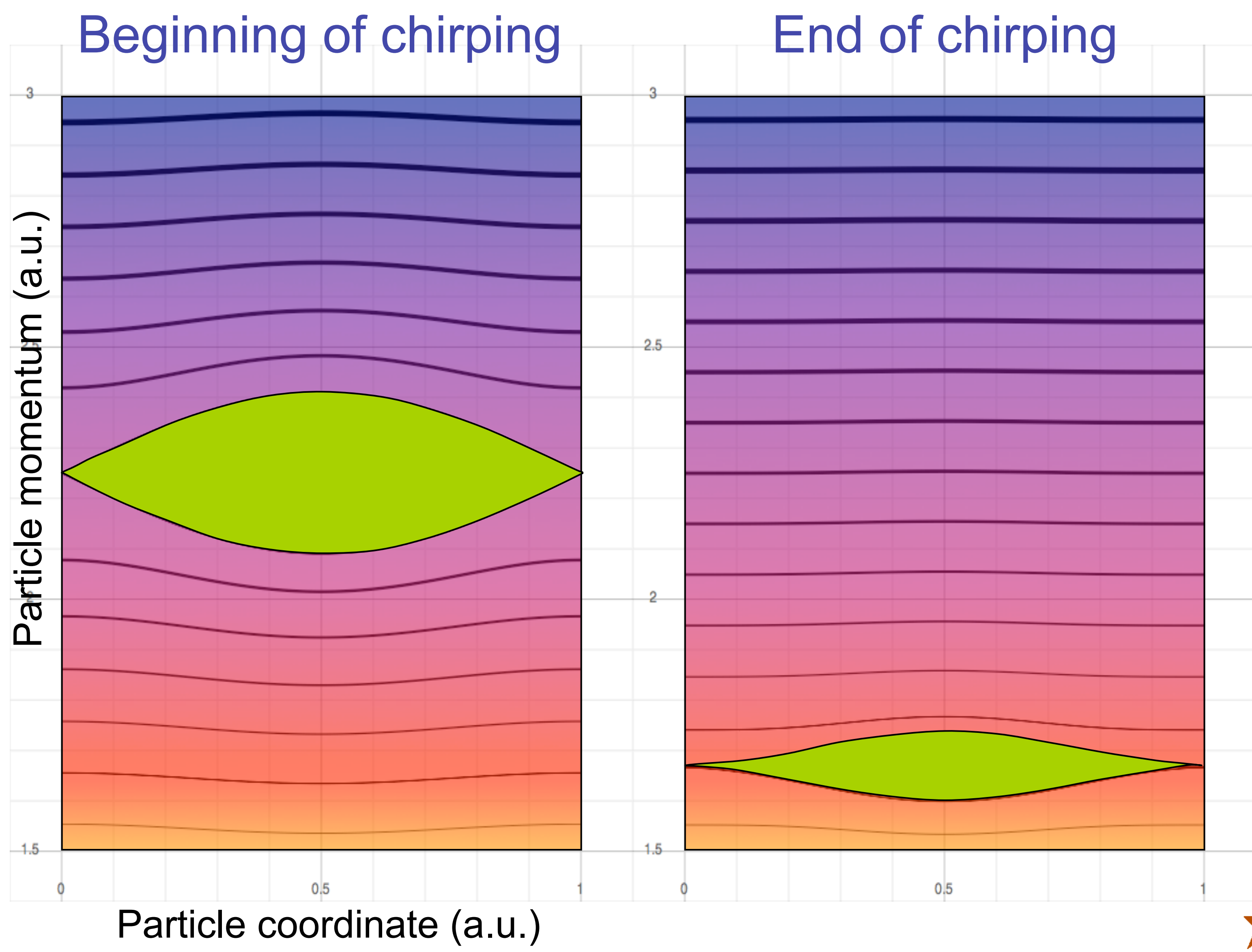
$$H(p;z;t) = \frac{p^2}{2m} - |e|\varphi\left[z - s(t);t\right]$$

- **Wave-frame Hamiltonian:**

$$H(p;x;t) = \frac{(p - m\dot{s})^2}{2m} - |e|\varphi(x;t) \qquad x = z - s(t)$$

- **Adiabatic invariants**
  - **Passing particles:** $J_\pm = \int_0^\lambda \left\{ m\dot{s} \pm \sqrt{2m\left[H + |e|\varphi(x;t)\right]} \right\} dx$
  - **Trapped particles:** $J_{trapped} = \oint \sqrt{2m\left[H + |e|\varphi(x;t)\right]}\, dx$

# Downward Drift of Phase Space Clump



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# BGK Mode Equation

- Electron potential energy $U$ has a given spatial period $\lambda$, a slowly varying shape, and a slowly varying phase velocity $\dot{s}$ :

$$U\left[z - s(t);t\right] \equiv -|e|\varphi$$

- Perturbed density of plasma electrons:

$$\delta n = n_0 U / m\dot{s}^2$$

- Local width of the separatrix:

$$\delta V = 2\sqrt{2/m}\sqrt{\left(U_{\max} - U\right)}$$

- Perturbed density of trapped electrons:

$$\delta n_t = \left[F_b(\dot{s}_0) - F_b(\dot{s})\right]2\sqrt{2/m}\left[\sqrt{\left(U_{\max} - U\right)} - \left\langle\sqrt{\left(U_{\max} - U\right)}\right\rangle\right]$$

- Nonlinear Poisson equation in the wave frame:

$$\frac{\partial^2 U}{\partial x^2} = -k^2 U - A(k)\left[\sqrt{\left(U_{\max} - U\right)} - \left\langle\sqrt{\left(U_{\max} - U\right)}\right\rangle\right]$$

$$k^2 \equiv \omega_p^2 / \dot{s}^2$$

$$A(k) \equiv 8\pi e^2\left[F_b(\dot{s}_0) - F_b(\dot{s})\right]\sqrt{2/m}$$



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# Power Balance and Chirping Rate

- Spatially periodic solution with variable phase velocity $\dot{s}$ :

$$U = \frac{m\dot{s}^2}{2}\left\{\frac{8\dot{s}\left[F_b(\dot{s}_0) - F_b(\dot{s})\right]}{3n_0\cos\alpha}\right\}^2 \left\{\frac{1+2\cos^2\alpha}{2} - \frac{3\sin 2\alpha}{4\alpha} - \left[\cos\alpha - \cos\left(\alpha\frac{2x}{\lambda} - \alpha\right)\right]^2\right\} \qquad \alpha(\dot{s}) \equiv \frac{\lambda\omega_p}{4\dot{s}}$$

- "Seed" profile of the wave: $U = \frac{m\dot{s}_0^2}{4}\left\{\frac{32}{3\pi^2}\frac{\gamma_L}{\omega_p}\right\}^2 \cos\left(\frac{2\pi x}{\lambda}\right) \qquad \frac{\gamma_L}{\omega_p} = \frac{\pi}{2}\frac{\dot{s}_0^2}{n_0}\frac{\partial F_b}{\partial \dot{s}_0}$

- Collisional dissipation in the bulk: $Q = \nu n_0 \frac{1}{m\dot{s}^2}\int_0^\lambda U^2 dx$

- Power release by the phase space clump:

$$P = -2m\dot{s}^2\lambda\left[F_b(\dot{s}_0) - F_b(\dot{s})\right]\left|\frac{8\dot{s}\left[F_b(\dot{s}_0) - F_b(\dot{s})\right]}{3n_0\cos\alpha}\right|\left(\frac{\sin\alpha}{\alpha} - \cos\alpha\right)\frac{d\dot{s}}{dt}$$

- Power balance condition ( $Q = P$ ) determines chirping rate.



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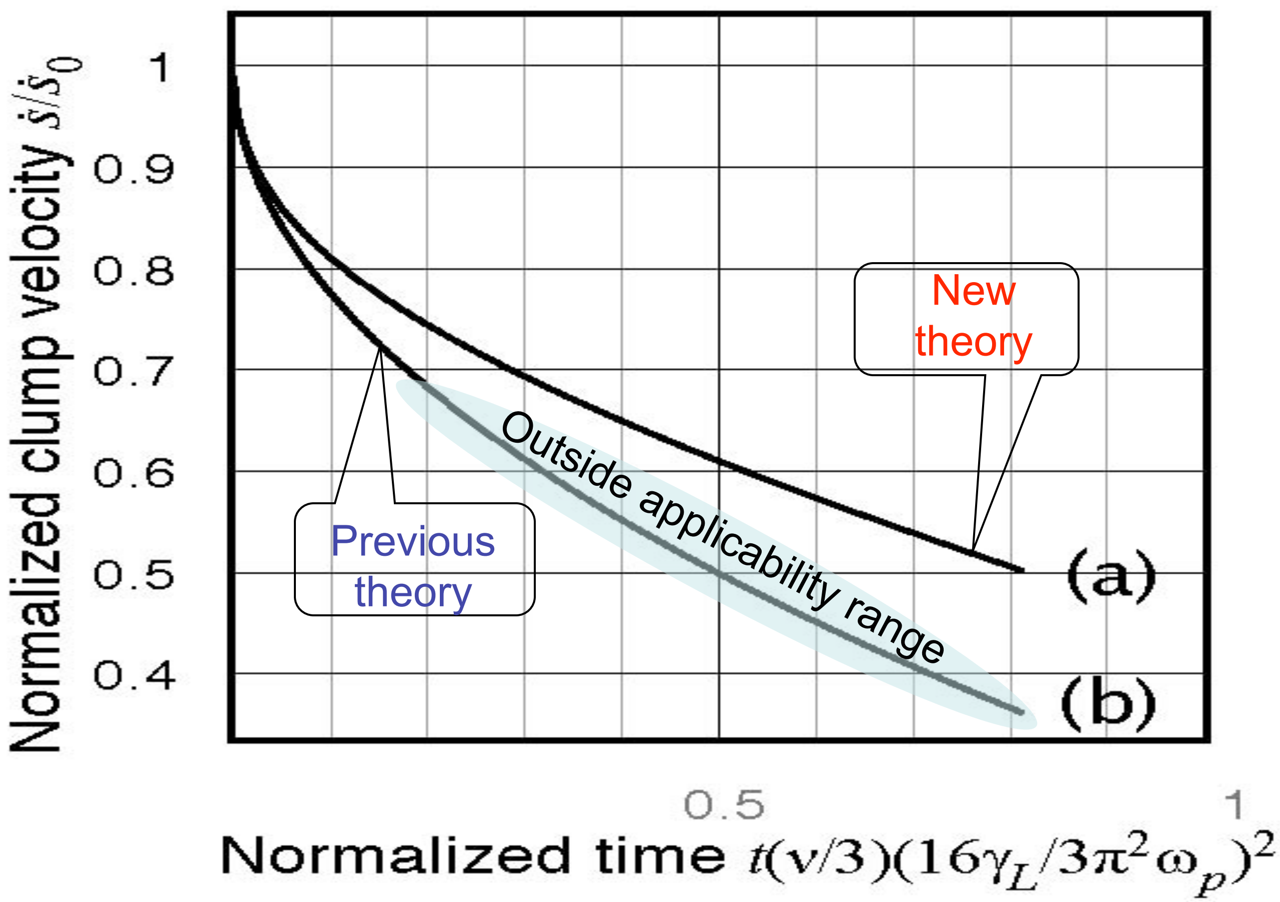
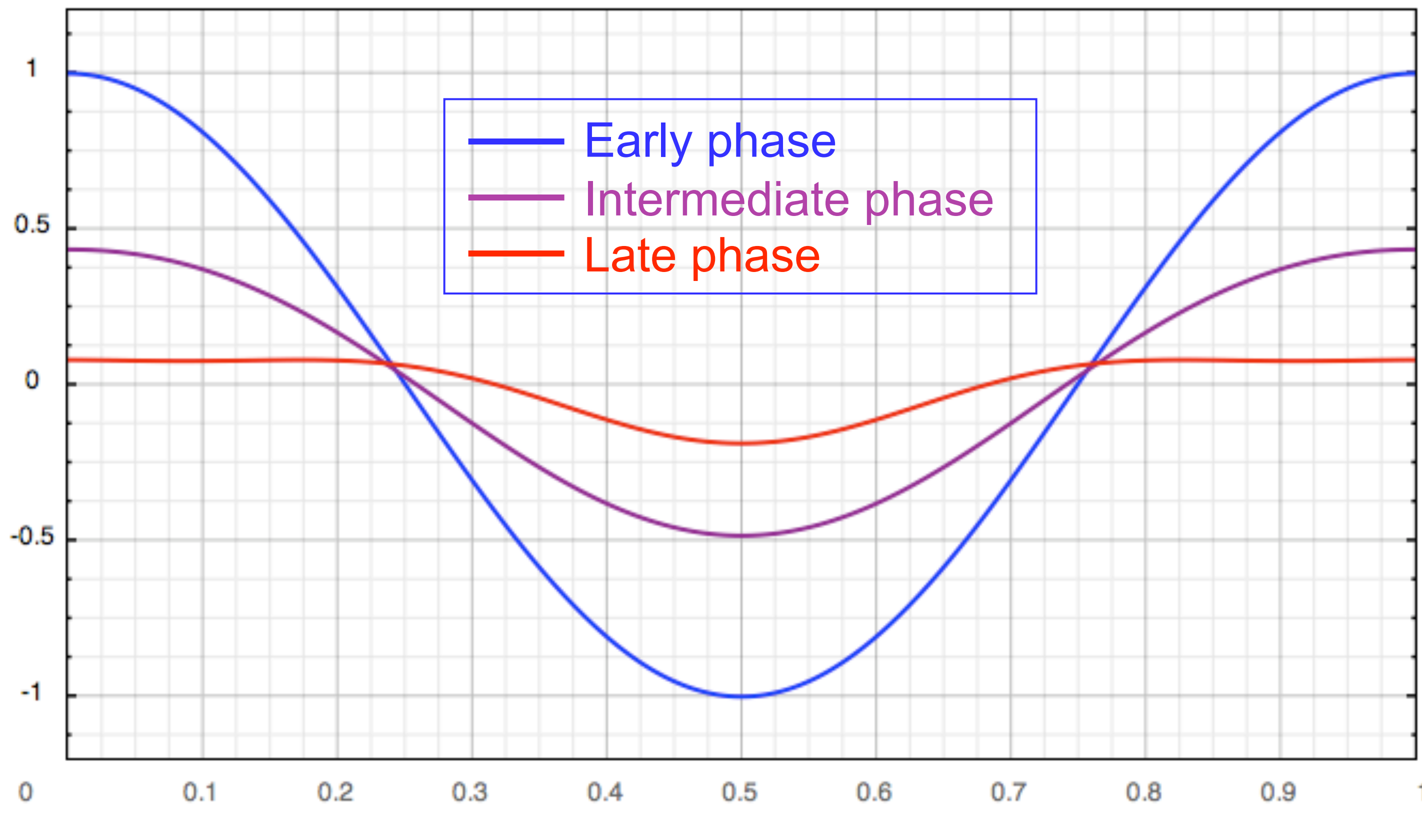
# Frequency Sweep Dynamics



# Wave Profile Evolution During Strong Chirping

- Wave amplitude decreases due to trapped particle leak
- The final profile has multiple harmonics



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# GENERALIZATION

## (how to apply the bump-on-tail model to fast particles in tokamaks)

# Wave-Particle Lagrangian

- **Perturbed guiding center Lagrangian:**

$$L = \sum_{\text{particles}} \left[ P_\vartheta \dot{\vartheta} + P_\varphi \dot{\varphi} - H\left(P_\vartheta; P_\varphi; \mu\right) \right] + \sum_{\text{modes}} \dot{\alpha} A^2 + 2\,\mathrm{Re} \sum_{\text{particles, modes, } l} A V_l\left(P_\vartheta; P_\varphi; \mu\right) \exp(-i\alpha - i\omega t + in\varphi + il\vartheta)$$

- **Dynamical variables:**
  - $P_\vartheta,\ \vartheta,\ P_\varphi,\ \varphi$ **are the action-angle variables for the particle unperturbed motion**
  - $A$ **is the mode amplitude**
  - $\alpha$ **is the mode phase**

- **Matrix element** $V_l\left(P_\vartheta; P_\varphi; \mu\right)$ **is a given function, determined by the linear mode structure**
- **Mode energy:** $W = \omega A^2$
- **Resonance condition:**

$$\Omega \equiv n\omega_\varphi\left(\mu; P_\varphi; E\right) - l\omega_\theta\left(\mu; P_\varphi; E\right) - s\omega_\psi\left(\mu; P_\varphi; E\right) - \omega = 0$$

The quantities $n$, $l$, and $s$ are integers with $s = 0$ for low-frequency modes.



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# Dynamical Equations

- Unperturbed particle motion is integrable and has canonical action-angle variables $I_i$ and $\xi_i$.
- Unperturbed particle motion is periodic in angles $\xi_1$, $\xi_2$, and $\xi_3$ .
- Single resonance approximation for the Hamiltonian:

$$\boxed{H = H_0(I) + 2\,\mathrm{Re}\left[A(t)V(I)\exp(i\xi - i\omega t)\right]}$$

- Kinetic equation with collisions included:

$$\boxed{\frac{\partial f}{\partial t} + \Omega(I)\frac{\partial f}{\partial \xi} - 2\,\mathrm{Re}\left[iA(t)\exp(i\xi - i\omega t)\right]\frac{\partial f}{\partial I} = \nu_{eff}^3\left(\partial\Omega/\partial I\right)^{-2}\frac{\partial^2 f}{\partial I^2}}$$

- Equation for the mode amplitude with background damping included:

$$\boxed{\frac{dA}{dt} = -\gamma_d A + \frac{i\omega}{G}\int d\Gamma V^* \exp(-i\xi + i\omega t) f}$$



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# Estimates of Resonance Width

- **Bump-on-tail kinetic equation** $\left(u \equiv k\mathrm{v} - \omega\right)$:

$$iuf_1 - \nu^3 \frac{\partial^2 f_1}{\partial u^2} = -\frac{ek}{2m}\hat{E}\frac{\partial F_0}{\partial u}$$

- **Compare collisional term to convective term to obtain** $\Delta u \sim \nu$ **.**

- **1-D reduction of collision operators for resonant particles in a tokamak:**

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial f}{\partial \mathbf{v}} = \underbrace{\left\langle \frac{\partial P_\phi}{\partial \mathbf{v}} \cdot \mathbf{D} \cdot \frac{\partial P_\varphi}{\partial \mathbf{v}} \right\rangle \left( \frac{\partial \Omega}{\partial P_\phi} \right)^2}_{\downarrow} \frac{\partial^2 f}{\partial \Omega^2}$$

$$\Delta\Omega^3_{\mathrm{Diff}} \quad \left(\mathrm{c.f.}\ \nu^3\right)$$

$$\frac{\partial}{\partial \mathbf{v}} \mathbf{b} f = \underbrace{\left\langle \frac{\partial P_\phi}{\partial \mathbf{v}} \cdot \mathbf{b} \right\rangle \left( \frac{\partial \Omega}{\partial P_\phi} \right)}_{\downarrow} \frac{\partial f}{\partial \Omega}$$

$$\Delta\Omega^2_{\mathrm{Drag}} \quad \left(\mathrm{c.f.}\ \alpha^2\right)$$



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# Linear Resonance Produces a BGK-mode

- **Each resonance can be treated separately when there is a large spacing between them**
- **The single-resonance Lagrangian has the form**

$$L = P_\theta \dot{\theta} + P_\varphi \dot{\varphi} - H\left(P_\theta; P_\varphi; \mu\right) + 2\,\mathrm{Re}\left[A V_l\left(P_\theta; P_\varphi; \mu\right)\exp(-i\omega t + in\varphi + il\theta)\right]$$

- **Additional Fourier harmonics build up when the linear mode becomes a BGK-mode, but periodicity of the perturbation is preserved.**
- **Generalization to the case of slowly evolving BGK-mode:**

$$L = P_\vartheta \dot{\vartheta} + P_\varphi \dot{\varphi} - H\left(P_\vartheta; P_\varphi; \mu\right) - U\left(\int \omega dt - n\varphi - l\vartheta; P_\theta; P_\varphi; \mu; t\right)$$

- **Potential energy $U$ is a periodic function of its first argument $\psi \equiv \int \omega dt - n\varphi - l\vartheta$ and a slow function of time.**



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# Resonant particles in a BGK-mode (contd.)

- Perturbed potential $U\left(\int \omega dt - n\varphi - l\vartheta; P_\theta; P_\varphi; \mu; t\right)$ describes a **traveling wave in phase space**

- Particle motion in the wave is one-dimensional due to conservation of magnetic moment $\mu$ and $lP_\varphi - nP_\theta$ .

- Similarly to the electrostatic problem, adiabatic invariant is conserved, except for the flow around the separatrix.

- The distribution function of trapped particles is constant within the separatrix.



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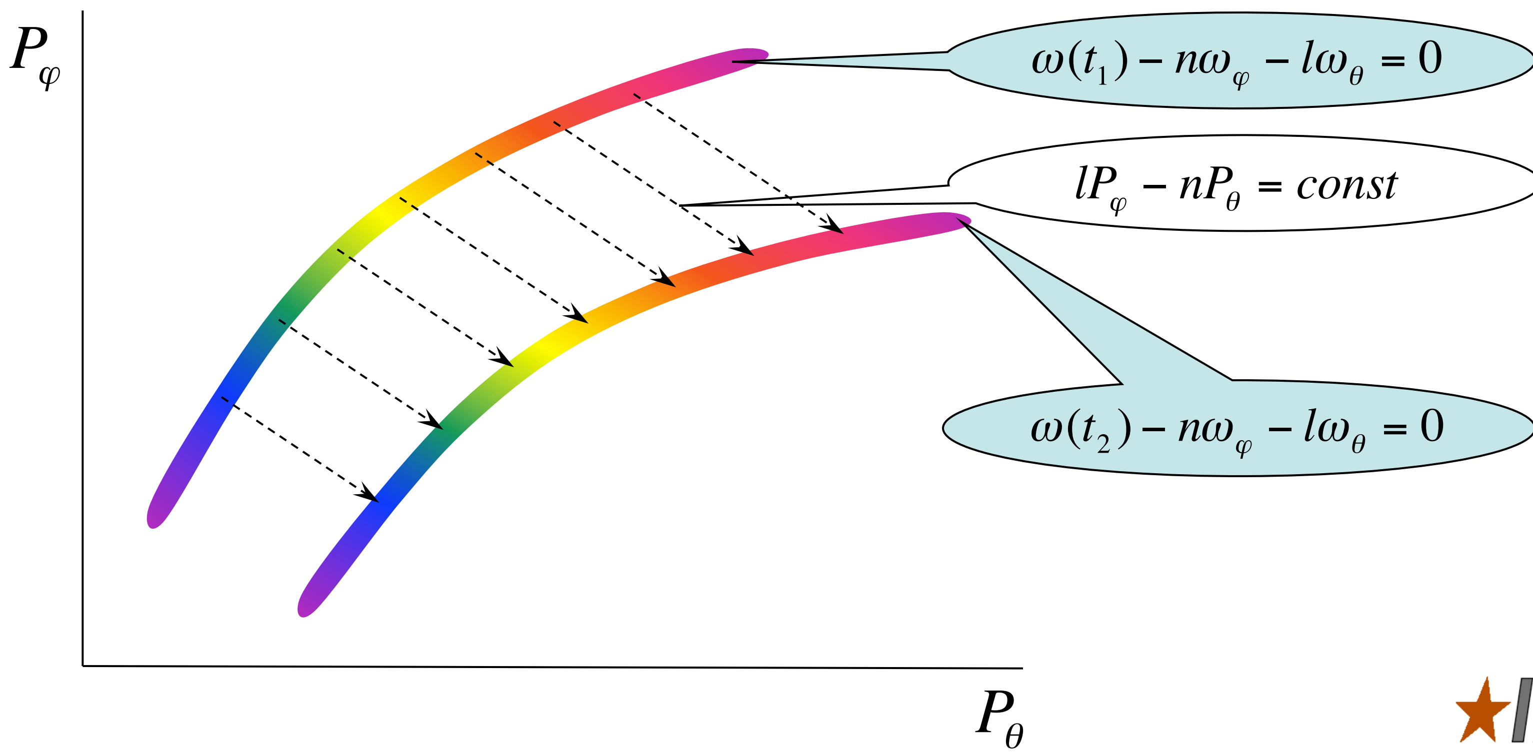
# Resonant Particle Convection

- The resonance carries trapped particles along the dotted lines.
- Initial distribution along the resonance (color coded) is preserved.
- Nonlinear particle response can be expressed analytically in terms of the perturbed fields (via waterbag approximation).



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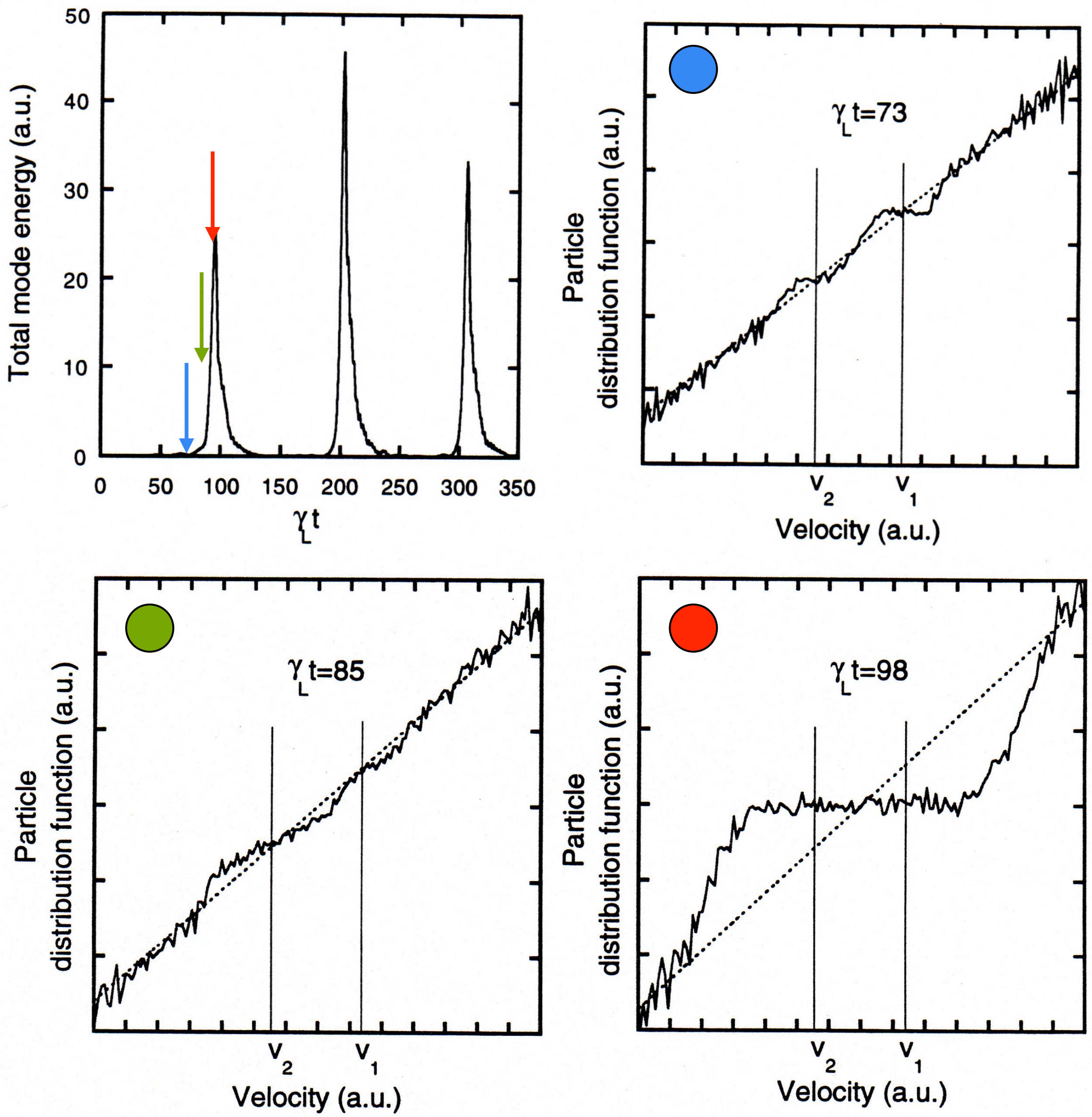
# MULTIPLE MODES AND GLOBAL TRANSPORT

## (mode bursts and profile stiffness)

# Effect of  Resonance Overlap




**The overlapped resonances release more free energy than the isolated resonances**

# Particle Transport Mechanisms of Interest

- **Neoclassical:** Large excursions of resonant particles (banana orbits) + collisional mixing

- **Convective:** Transport of phase-space holes and clumps by modes with frequency chirping

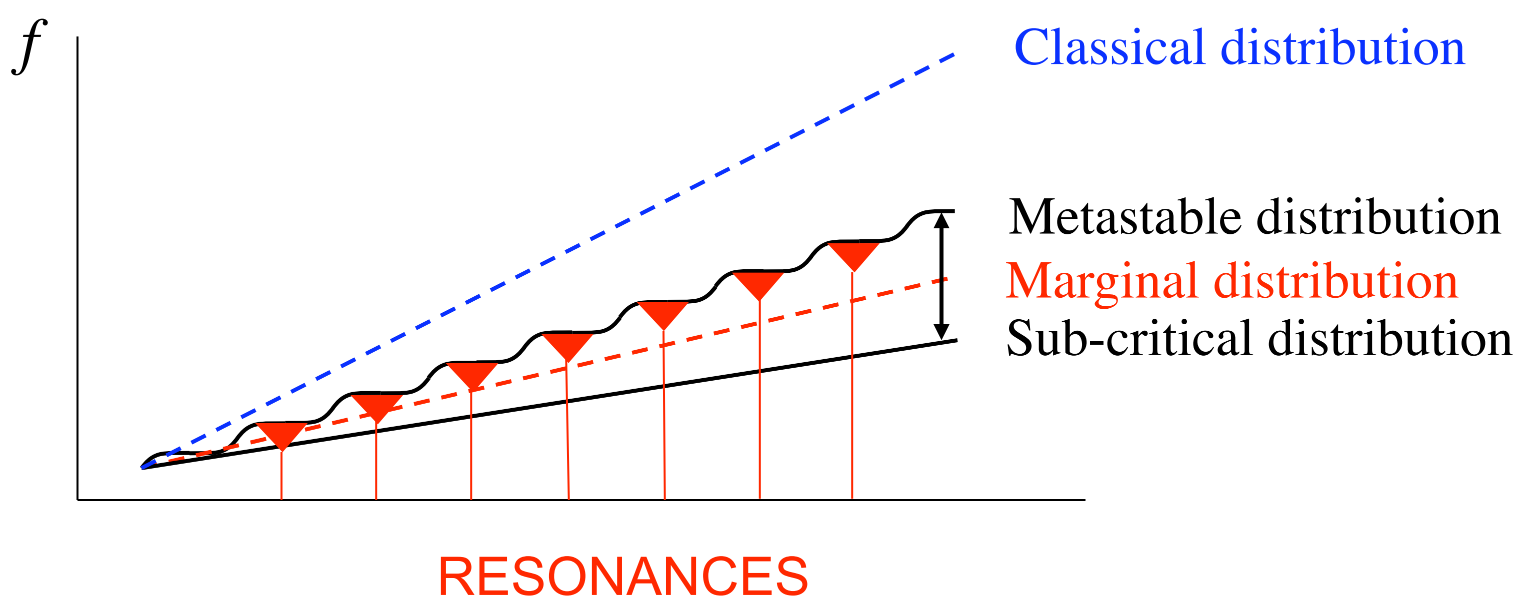
- **Quasilinear :** Phase-space diffusion over a set of overlapped resonances

**Important Issue:** Individual resonances are narrow. How can they affect every particle in phase space?

# Intermittent Quasilinear Diffusion

**A weak source (with insufficient power to overlap the resonances) is unable to maintain steady quasilinear diffusion**

**Bursts occur near the marginally stable case**



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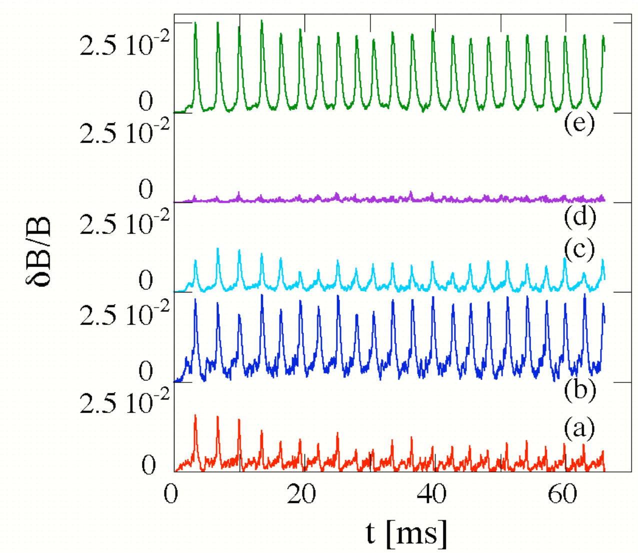
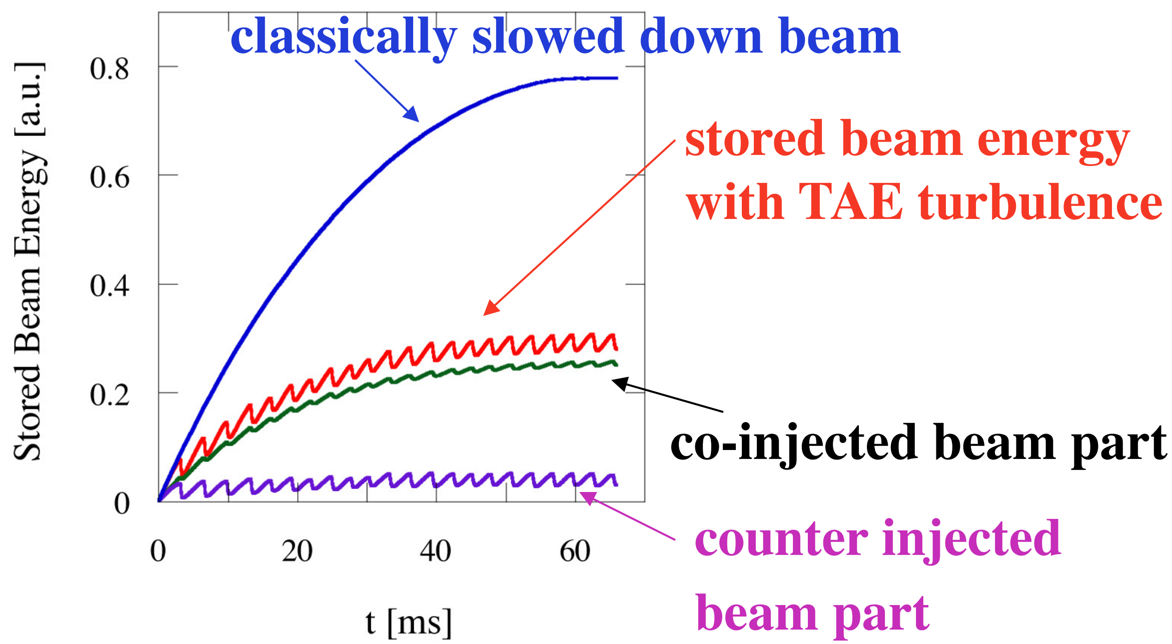
# Simulation of Intermittent Losses




Numerical simulations of Toroidal Alfvén Eigenmode (TAE) bursts with parameters relevant to TFTR experiments have reproduced several important features:

- **synchronization of multiple TAEs**
- **timing of bursts**
- **stored beam energy saturation**

*Y. Todo, et al., Phys. Plasmas **10**, 2888 (2003)*

# Issues in Modeling Global Transport

- **Reconciliation of mode saturation levels with experimental data**
  - Simulations reproduce bursty losses and particle accumulation level
  - Wave saturation amplitudes appear to be larger than the experimental values (however, this discrepancy may not affect the fast particle pressure profiles)
- **Edge effects in fast particle transport**
  - Sufficient to suppress modes locally near the edge
  - Need better description of edge plasma parameters
- **Creation of transport barriers for fast particles**



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# CONCLUDING REMARKS

- All particles are equal but resonant particles are more equal than others.

- Near-threshold kinetic instabilities in fusion-grade plasmas exhibit rich but comprehensible non-linear dynamics of very basic nature.

- Nonlinear physics offers interesting diagnostic opportunities associated with bifurcations and coherent structures.

- Energetic particle driven turbulence is prone to intermittency that involves avalanche-type bursts in particle transport.